# RELATIONS BETWEEN NONSTEADY AND STEADY TEMPERATURE FIELDS

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Relations are established between nonsteady temperature fields and their limiting states, steady fields, for linear and some nonlinear boundary problems.

## I. Linear Case

Suppose that it is required to determine U(M, t), satisfying the condition

$$U_t = a^2\Delta U;\quad M\in D,\ t>0,\ U(M,\ 0)=\varphi(M),\ \alpha\frac{\partial U}{\partial n}+U=0,\ M\in G, \tag{1}$$

where $a$ and $\alpha$ are constants; $\varphi(M)$, a known function; D, an n-dimensional region with boundary G. The solution of this problem is written in the form

$$U(M,\ t)=\sum_{n=1}^{\infty} c_nV_n\exp(-a^2\lambda_n t),\quad c_n=\|V_n\|^{-2}\int_D \varphi(P)\,V_n(P)\,d\sigma, \tag{2}$$

where $\lambda_n$ are eigenvalues; $V_n$ are the eigenfunctions of the corresponding Sturm–Liouville problem

$$\Delta V+\lambda V=0,\ M\in D,\ \alpha\frac{\partial V}{\partial n}+V=0,\quad M\in G. \tag{3}$$

As is known, Eq. (2) is a series converging uniformly in t. Term-by-term integration gives

$$\int_0^t U(M,\ \tau)\,d\tau=\sum_{n=1}^{\infty} c_nV_n\int_0^t \exp(-a^2\lambda_n\tau)\,d\tau.$$

Calculating the integral on the right-hand side and passing to the limit as $t\to\infty$, it is found that

$$F(M)\equiv\lim_{t\to\infty}\int_0^t U(M,\ \tau)\,d\tau=\sum_{n=1}^{\infty}\frac{c_nV_n}{\lambda_n a^2}. \tag{4}$$

Using Eq. (3), it is found that F(M) satisfies the following conditions

$$\begin{gathered} a^2\Delta F(M)=-\varphi(M),\quad M\in D,\\ \alpha\frac{\partial F(M)}{\partial n}+F(M)=0,\quad M\in G. \end{gathered} \tag{5}$$

Thus, the limiting relation between the solution of the heat-conduction equation (4) and the solution of Poisson's equation (5) is obtained.

Now suppose that it is required to determine U(M, t)

$$\begin{gathered} U_t=a^2\Delta U+f(M,\ t),\quad M\in D,\ t>0,\ U(M,\ 0)=0,\\ \alpha\frac{\partial U}{\partial n}+U=0,\quad M\in G. \end{gathered} \tag{6}$$

The solution of Eq. (6) takes the form

$$U(M,\ t)=\sum_{n=1}^{\infty} V_n(M)\,f_n(\tau^*)\int_0^t \exp(-a^2\lambda_n(t-\tau))\,d\tau,\quad \tau^*\in[0,\ t],$$

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where

$$f_n(t) = \|V_n\|^{-2} \int_D f(P, t) V_n(P) d\sigma, \quad f(M, t) = \sum_{n=1}^{\infty} f_n(t) V_n(M).$$

If $t \to \infty$, then

$$F(M) \equiv \lim_{t \to \infty} U(M, t) = \sum_{n=1}^{\infty} V_n(M) f_n(\infty) \frac{1}{a^2 \lambda_n}, \tag{7}$$

and hence

$$a^2 \Delta F = \sum_{n=1}^{\infty} \frac{1}{a^2 \lambda_n} [a^2 \Delta V_n f_n(\infty)] = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} [-\lambda_n V_n f_n(\infty)] = -\sum_{n=1}^{\infty} f_n(\infty) V_n(M) = -\lim_{t \to \infty} f(M, t) = -\psi(M).$$

Thus, the limiting steady field is described by the conditions

$$a^2 \Delta F(M) = -\psi(M), \quad M \in D, \quad \alpha \frac{\partial F}{\partial n} + F = 0, \quad M \in G. \tag{8}$$

Note that, if $f(M, t)$ in Eq. (6) is independent of $t$, i.e., $f = f(M)$, then $f_n(t) = \text{const} = f_n(\infty)$.

## II. Nonlinear Case

Next, the relation between nonsteady and steady fields is investigated in the case when the thermal conductivity depends on the temperature. It is assumed that the small parameter $\varepsilon$ appears in the equation. Let

$$U_t = \operatorname{div} [\lambda(\varepsilon U) \operatorname{grad} U] + f(M, t), \quad U(M, 0) = \varphi(M), \quad M \in D,$$
$$\alpha \frac{\partial U}{\partial n} + U = 0, \quad M \in G. \tag{9}$$

The solution of this problem is found in the form of a power series in $\varepsilon$ [1, 2]

$$U(M, t) = U_0(M, t) + \varepsilon U_1(M, t) + \ldots + \varepsilon^k U_k(M, t) + \ldots$$

under the condition that it converges. The functions $U_k(M, t)$ are solutions of the following problems

$$\frac{\partial U_0}{\partial t} = \lambda(0) \Delta U_0 + f_0(M, t), \quad U_0(M, 0) = \varphi(M), \quad \alpha \frac{\partial U_0}{\partial n} + U_0 = 0; \tag{10}$$

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$$\frac{\partial U_k}{\partial t} = \lambda(0) \Delta U_k + f_k(M, t, U_0, \ldots U_{k-1}),$$
$$U_k(M, 0) = 0, \quad \alpha \frac{\partial U_k}{\partial n} + U_k = 0, \tag{11}$$

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The solution of the problem in Eq. (10) is sought in the form of a sum of two functions $U_0 = P + Q$, satisfying the conditions

$$\left.\begin{array}{l} P_t = \lambda(0) \Delta P + f_0, \\ P(M, 0) = 0, \\ \alpha \dfrac{\partial P}{\partial n} + P = 0, \end{array}\right\} \quad \left.\begin{array}{l} Q_t = \lambda(0) \Delta Q \\ Q(M, 0) = \varphi(M), \\ \alpha \dfrac{\partial Q}{\partial n} + Q = 0, \end{array}\right\} \begin{array}{l} M \in D, \\ \\ M \in G. \end{array} \tag{12}$$

In view of Eqs. (7) and (8), $\lim_{t \to \infty} P(M, t) = F_0(M)$, and

$$\lambda(0) \Delta F_0 = -f_0, \quad M \in D, \quad \alpha \frac{\partial F_0}{\partial n} + F_0 = 0, \quad M \in G.$$

Taking Eqs. (4) and (5) into account yields

$$\lim_{t\to\infty}\int_0^t Q(M,\tau)\,d\tau = \Phi(M),$$

and

$$\lambda(0)\,\Delta\Phi = -\varphi(M), \quad \alpha\frac{\partial\Phi}{\partial n} + \Phi = 0.$$

Note that $Q(M,t)\underset{t\to\infty}{\to}0$ . In view of Eqs. (7) and (8), the solution of Eq. (11) satisfies the relation

$$\lim_{t\to\infty} U_k(M,t) = F_k(M),$$

while the function $F_k(M)$ is a solution of the problem

$$\lambda(0)\,\Delta F_k(M) = -\overline{f}_k(M) = \lim_{t\to\infty} f_k, \quad \alpha\frac{\partial F_k}{\partial n} + F_k = 0, \; M\in G.$$

Thus, the following conclusions may be derived:

a) as $t\to\infty$, the solution $U(M,t)$ of Eq. (9) tends to some function

$$W(M) = \lim_{t\to\infty} U(M,t) = \lim_{t\to\infty}\left(P + Q + \sum_{k=1}^{\infty}\varepsilon^k U_k\right) = \sum_{k=0}^{\infty}\varepsilon^k F_k(M),$$

where $F_k(M)$ is a solution of problems of the form in Eqs. (10) and (11);

b) the function $W(M)$ satisfies the conditions

$$\operatorname{div}[\lambda(\varepsilon W)\operatorname{grad} W] = -\psi(M), \; M\in D, \quad \alpha\frac{\partial W}{\partial n} + W = 0, \quad M\in G, \tag{13}$$

where $\psi(M) = \lim_{t\to\infty} f(M,t)$ .

c) in those cases where it is necessary to know the thermal state of the body at sufficiently high t, the simpler boundary problem in Eq. (13) may be solved instead of Eq. (9).

As an example, consider the problem

$$V_t = [\{1 + 4\varepsilon(1+V)^3\}V_x]_x + f(x,t), \quad V(x,0) = 0,$$
$$V_x(0,t) = 0, \quad V_x(1,t) = 2[1-\exp(-\alpha t)].$$

Here

$$f(x,t) = \alpha x^2\exp(-\alpha t) - 2[1-\exp(-\alpha t)] - 8[1-\exp(-\alpha t)]\{1 + x^2\times$$
$$\times[1-\exp(-\alpha t)]\}^2\{1 + 7x^2[1-\exp(-\alpha t)]\}\,\varepsilon.$$

It is found that

$$\psi(x) = \lim_{t\to\infty} f(x,t) = -2 - 8\varepsilon(1+x^2)^2(1+7x^2).$$

The equation of the steady state is written as follows

$$[\{1+4\varepsilon(1+W)^3\}W_x]_x = 2 + 8\varepsilon(1+x^2)^2(1+7x^2),$$
$$W_x(0) = 0, \quad W_x(1) = 2. \tag{14}$$

The problem in Eq. (14) has a unique positive solution $W = x^2$. It is clear from a comparison of the accurate solution of the problem

$$V(x,t) = x^2[1-\exp(-\alpha t)]$$

and the function $W(x)$ that

$$W(x) = \lim_{t\to\infty} V(x,t).$$

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# REGULARIZATION OF INVERSE PROBLEMS BY THE SCHEME OF PARTIAL MATCHING WITH ELEMENTS OF A SET OF OBSERVATIONS

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The problem of determining the thermophysical properties by means of a discrete set of observations on the temperatures of the test object given with measurement errors is examined.

The investigation of complex processes by using inverse problems has attracted considerable attention lately. Their solution is associated with certain singularities, particularly the influence of errors in the initial data on the desired solution. As is known from [1], in such cases it is necessary to limit the domain of the allowable solutions and to match the measurement errors. Since a number of stabilizing functionals with the same problem can be set in correspondence and different norms for the deviation from the quantities observed can be selected, then it is interesting to determine those among them which will permit, for sufficiently general assumptions about the desired quantities, obtaining the most exact solutions under conditions of unimprovable observations for a broad range of measurement errors. In addition, the question of selecting the method of matching the observations occurs in the solution of applied ill-posed problems. One condition that establishes a relation between the accuracy of the solution and the measurement error [2] is used in the widespread problem, in practice, of restoring the thermal flux. This condition expresses the total error in all observations for measurements executed at several points. However, one condition can turn out to be inadequate to determine several parameters of a model that is characteristic for the inverse coefficient problems, while taking total account of the errors results in a loss in accuracy of the solution of the inverse problem [3]. This paper is devoted to investigating the properties of the regularized solution of an inverse coefficient problem for the nonlinear heat-conduction equation as a function of the degree of limitation of the domain of admissible solutions, the form of the observation error estimate, and the methods of matching them.

In the domain $Q = \{(x, t): 0 < x < 1, 0 < t < T\}$ we examine the one-dimensional heat-conduction equation

$$a_1 \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_2 \frac{\partial u}{\partial x}\right) + f(x, t) \tag{1}$$

for which the initial and boundary conditions assuring uniqueness and stability in the determination of the function u(x, t) for given values of the specific heat $a_1(u)$ and the heat conductivity $a_2(u)$ and any T > 0 are assumed known.

Let us also assume that at m points of space, and for each of n times of the domain Q observation results are given

$$u_{ij}^{\delta} = \bar{u}(x_i, t_j) + \varepsilon_{ij}, \quad i = \overline{1, m}, \ j = \overline{1, n}, \tag{2}$$

with a known magnitude of the deviation norm

$$\delta_i^2 = \sum_{j=1}^{n} (u_{ij}^{\delta} - \bar{u}_{ij})^2, \quad i = \overline{1, m}, \tag{3}$$

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